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Dipole field solution of Maxwell's equations in the Schwarzschild metric

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Abstract. The dipole solution of Maxwell's equations in the Schwarzschild metric is found as a convergent series for $r > 2m$. The validity of this representation by series is proved by a complete study of its convergence and differentiability.

1. Introduction

In the Newman and Penrose (1962) formalism, the electromagnetic field is characterized by three complex quantities Φ_0, Φ_1 and Φ_2. When we consider the field of an electric dipole in the Schwarzschild metric, the angular dependence is

$$\Phi_0 = \phi_0(u, r)\eth Y_1^0(\theta, \varphi), \qquad \Phi_1 = \phi_1(u, r) Y_1^0(\theta, \varphi) \quad \text{and} \quad \Phi_2 = \phi_2(u, r)\bar{\eth} Y_1^0(\theta, \varphi).$$

Papapetrou (1975) has shown that $A = r\phi_2$ satisfies the partial differential equation

$$\frac{\partial^2}{\partial u \partial r}A - \frac{1}{2}\frac{\partial^2}{\partial r^2}A + \frac{m}{r}\frac{\partial^2}{\partial r^2}A - \frac{1}{r}\frac{\partial}{\partial r}A + \frac{2m}{r^2}\frac{\partial}{\partial r}A + \frac{1}{r^2}A = 0. \tag{1}$$

The solution of this equation completely determines the solution of Maxwell's equations:

$$\phi_2 = \frac{1}{r}A, \qquad \phi_1 = -\sqrt{2}\frac{\partial}{\partial r}A \qquad \text{and} \qquad \phi_0 = -\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}A\right). \tag{2}$$

To solve equation (1), we seek a solution in the form:

$$A(u, r) = \ddot{p}(u) + \dot{p}(u)\frac{1}{r} + \tfrac{1}{2}p(u)\frac{1}{r^2} + a(u, r) \tag{3}$$

where the dipole moment p is a function of the retarded time.

In the Minkowski metric, the function a would vanish and the expression (3) would be the solution of the equation (1) with $m = 0$ and without incoming radiation.

The time-independent solution of the equation (1) corresponding to a static dipole can be expressed in closed form. This solution can be given for $r > 2m$ by a convergent series (Papapetrou 1975). For a dipole moment $p = 1$, we have

$$a(r) = m\frac{a_1}{r^3} + \ldots + m^n\frac{a_n}{r^{n+2}} + \ldots \tag{4}$$

where the relation between two consecutive coefficients is

$$a_n = \frac{2(n+1)}{(n+3)} a_{n-1}, \qquad n \geqslant 2 \tag{5}$$

with $a_1 = \frac{1}{2}$.

In the general case, the function a can be given for $r > 2m$ by a convergent series:

$$a(u, r) = m a_1(u, r) + \ldots + m^n a_n(u, r) + \ldots . \tag{6}$$

Papapetrou (1975) has determined the coefficients a_1 and a_2. We shall give here an integral expression for a_n. Moreover, we shall prove that the series (6) satisfies the convergence criteria which are necessary in order that (6) represents the solution of the equation (1). Our results for the dipole electromagnetic field are more complete than some similar results given by Bardeen and Press (1973) in a general case.

We shall consider first the case in which $p(u)$ is a step function $S(u-u_0)$. The solution which we shall determine for this case, will allow us to derive the solution for the case of an arbitrarily given $p(u)$, because of the linearity of the equation (1).

2. Solution when the dipole moment is a step function

2.1. Determination of the terms of the series

After substituting the expression (6) into equation (1), we have the system of the equations:

$$\frac{\partial^2}{\partial u \partial r} a_1 - \frac{1}{2}\frac{\partial^2}{\partial r^2} a_1 - \frac{1}{r}\frac{\partial}{\partial r} a_1 + \frac{1}{r^2} a_1 = -\frac{S(u-u_0)}{r^5} \tag{7}$$

$$\frac{\partial^2}{\partial u \partial r} a_n - \frac{1}{2}\frac{\partial^2}{\partial r^2} a_n - \frac{1}{r}\frac{\partial}{\partial r} a_n + \frac{1}{r^2} a_n = -\frac{1}{r}\frac{\partial^2}{\partial r^2} a_{n-1} - \frac{2}{r^2}\frac{\partial}{\partial r} a_{n-1}, \qquad n \geqslant 2. \tag{8}$$

Papapetrou's results (1975) suggest that

$$a_n(u, r) = \frac{S(u-u_0)}{r^{n+2}} f_n(y) \qquad \text{with } y = \frac{u-u_0}{2r}, \qquad y \geqslant 0. \tag{9}$$

If the functions a_n are solutions of the equations (7) and (8), then the functions f_n satisfy first of all the system of differential equations:

$$y(1+y) f_1'' + (4+6y) f_1' + 4 f_1 = 2 \tag{10}$$

$$y(1+y) f_n'' + [(n+3) + 2(n+2)y] f_n' + n(n+3) f_n = 2 v_{n-1}(y), \qquad n \geqslant 2 \tag{11}$$

where we set

$$v_n(y) = y^2 f_n'' + 2(n+2) y f_n' + (n+1)(n+2) f_n. \tag{12}$$

Additionally the functions f_n must satisfy

$$f_n(0) = 0 \qquad \text{and} \qquad f_n'(0) < \infty. \tag{13}$$

The function f_n can be obtained with the help of the function f_{n-1} by determining the regular solution at the point $y = 0$ of the equation (11). The function f_1 has been given by Papapetrou (1975):

$$f_1(y) = \frac{1}{2} - \frac{1}{2(1+y)}. \tag{14}$$

For f_n, we find

$$f_n(y) = \frac{1}{(1+y)^n}\left[12\int_0^y \frac{(1+t)^{n-1}}{t^{n+3}}\left(\int_0^t x^{n+2} f_{n-1}(x)\,\mathrm{d}x\right)\mathrm{d}t + 2\int_0^y (1+t)^{n-2}[-2t+(n-3)]f_{n-1}(t)\,\mathrm{d}t + 2y(1+y)^{n-1}f_{n-1}(y)\right],$$
$$n \geqslant 2. \tag{15}$$

The functions f_n are indefinitely differentiable and their derivatives are regular at the point $y = 0$. In particular, we have

$$f_n^{(p)}(0) = 0 \qquad \text{for } 1 \leqslant p \leqslant n-1. \tag{16}$$

The equation (11) possesses a property of particular interest. If we differentiate the expression (11), we have for f'_n an equation of the same form with $(n+1)$ instead of n and f'_{n-1} instead of f_{n-1}.

This property allows us to write expressions similar to (14) and (15) for the derivatives of f_n:

$$f_1^{(p)}(y) = -\tfrac{1}{2}(-1)^p p!\frac{1}{(1+y)^{p+1}} \tag{17}$$

$$f_n^{(p)}(y) = \frac{1}{(1+y)^{n+p}}\left[12\int_0^y \frac{(1+t)^{n+p-1}}{t^{n+p+3}}\left(\int_0^t x^{n+p+2} f_{n-1}^{(p)}(x)\,\mathrm{d}x\right)\mathrm{d}t + 2\int_0^y (1+t)^{n+p-2}[-2t+(n+p-3)]f_{n-1}^{(p)}(t)\,\mathrm{d}t + 2y(1+y)^{n+p-1}f_{n-1}^{(p)}(y) + f_n^{(p)}(0)\right] \qquad n \geqslant 2. \tag{18}$$

2.2. *Properties of the terms of the series*

We introduce the notation

$$\|f(y)\| = \sup|f(y)| \qquad \text{for } y \in [0, \infty].$$

We want to show that for all n, the quantity $\|(1+y)^{p+1}f_n^{(p)}(y)\|$ exists for $p \geqslant 1$. This is true for $n = 1$ according to expression (17). Suppose the property is true for $n-1$. The quantity $(1+y)^{p+1}f_n^{(p)}(y)$ can be computed from the formula (18). Under the integral sign $(1+y)^{p+1}|f_{n-1}^{(p)}(y)|$ is bounded by $\|(1+y)^{p+1}f_{n-1}^{(p)}(y)\|$. Thus, we have the inequality:

$$(1+y)^{p+1}|f_n^{(p)}(y)| \leqslant \frac{1}{(1+y)^{n-1}}\left[12\int_0^y \frac{(1+t)^{n+p-1}}{t^{n+p+3}}\left(\int_0^t \frac{x^{n+p+2}}{(1+x)^{p+1}}\mathrm{d}x\right)\mathrm{d}t + 2\int_0^y (1+t)^{n-3}[2t+(n+p-3)]\,\mathrm{d}t + 2y(1+y)^{n-2}\right]\|(1+y)^{p+1}f_{n-1}^{(p)}(y)\| + |f_n^{(p)}(0)| \qquad n \geqslant 2, p \geqslant 1. \tag{19}$$

It follows that $\|(1+y)^{p+1}f_n^{(p)}(y)\|$ exists for $p \geqslant 1$. Consequently, we shall have

$$|f_n^{(p)}(y)| \leqslant \frac{\|(1+y)^{p+1}f_n^{(p)}(y)\|}{(1+y)^{p+1}}, \qquad y \geqslant 0, p \geqslant 1. \tag{20}$$

When $n \geqslant p+1$, integration in the expression (19) leads to the inequality:

$$\|(1+y)^{p+1}f_n^{(p)}(y)\| \leqslant \left\{ 2+\mathrm{O}\left(\frac{1}{n}\right) \right\} \|(1+y)^{p+1}f_{n-1}^{(p)}(y)\|, \qquad p \geqslant 1. \tag{21}$$

We thus find the basic property:

$$\lim_{n\to\infty} \frac{\|(1+y)^{p+1}f_n^{(p)}(y)\|}{\|(1+y)^{p+1}f_{n-1}^{(p)}(y)\|} \leqslant 2, \qquad p \geqslant 1. \tag{22}$$

It is possible to complete the study of the functions f_n and to show that the f_n, and also the v_n, are positive and monotonic. Moreover, it is easy to see that

$$\lim_{y\to\infty} f_n(y) = a_n.$$

Thus, the function f_n is bounded by a_n. Therefore by comparison with the convergent series (4), the series (6), whose terms are given by the formula (9), converges for $r > 2m$.

We remark that for r fixed when $u \to \infty$ the field tends to a limit which is the static field of a dipole moment $p = 1$.

However, we shall see that it is possible to prove that the series (6) represents the solutions of equation (1) by using only the basic property (22). In order to arrive at this proof we have to consider the solution of equations (1) for the case of a dipole moment which is a delta function.

3. Solution when the dipole moment is a Dirac delta function

It is clear that the solution corresponding to the dipole moment $\delta(u-u_0)$ can be obtained by differentiating with respect to $-u_0$ the solution determined in § 2 for the dipole moment $S(u-u_0)$. The final expression is then found to be

$$G(u-u_0, r) = S(u-u_0)H(u-u_0, r)$$

with

$$H(u,r) = \sum_{n=1}^{\infty} \frac{m^n}{2r^{n+3}} f_n'\left(\frac{u}{2r}\right). \tag{23}$$

Using the basic properties (20) and (22), we see now at once that the series H, and all its formal partial derivatives with respect to u and r, are absolutely and uniformly convergent in any domain $r \geqslant r_1 > 2m$ with any r_1.

Thus, all partial derivatives of the function H can be obtained by differentiation term by term. Moreover we can change the order of the terms. This allows us to verify that the function G given by (23) is the required function a.

4. Solution for any dipole moment

Remembering that

$$p(u) = \int_{-\infty}^{+\infty} p(u_0)\delta(u-u_0)\,\mathrm{d}u_0$$

we see that for any dipole moment $p(u)$, the function a will be given by the formal expression:

$$a(u,r) = \int_{-\infty}^{+\infty} p(u_0)G(u-u_0, r)\,\mathrm{d}u_0 \tag{24}$$

valid for $r > 2m$.

Let us assume that for all u the function $|p(u')|$ is bounded by a constant K_u if $u' \leqslant u$.

The basic formulae (20) and (22) show that the function H and its derivatives are in a neighbourhood of any point (u_2, r_2) bounded by an integrable function.

This result and the assumption on $p(u)$ show that the function a given by (24) exists and is continuous; moreover we have the formulae:

$$\frac{\partial a}{\partial r} = \int_{-\infty}^{u} p(u_0)\frac{\partial H}{\partial r}(u-u_0, r)\,\mathrm{d}u_0 \tag{25}$$

$$\frac{\partial a}{\partial u} = \frac{1}{4}\frac{m}{r^4}p(u) + \int_{-\infty}^{u} p(u_0)\frac{\partial H}{\partial u}(u-u_0, r)\,\mathrm{d}u_0. \tag{26}$$

We can again differentiate the expressions (25) and (26) with respect to r. These differentiations under the integral sign show that a given by (24) is the required function a, since G is the function a for a dipole moment $\delta(u-u_0)$.

It is easy to see that if the dipole moment p is a C^k function $k \geqslant 0$, with $|p(u)|$ bounded in the limit $u \to -\infty$, then the function a given by the formula (24) is a C^{k+1} function.

We remark that we can change the order of integration and summation in the formula (24). Consequently, we have

$$a(u,r) = \sum_{n=1}^{\infty} \frac{m^n}{2r^{n+3}} \int_{-\infty}^{u} p(u_0) f_n'\left(\frac{u-u_0}{2r}\right)\mathrm{d}u_0, \qquad r > 2m. \tag{27}$$

Again from the basic properties (20) and (22), the series (27) and all its partial derivatives with respect to u and r, are absolutely and uniformly convergent in any domain $r \geqslant r_1 > 2m$ with any r_1.

5. Conclusion

We have determined the component ϕ_2 of the field of an electric dipole in the Schwarzschild metric. The two other components are then calculated from (2):

$$\phi_1 = \sqrt{2}\left(\frac{\dot{p}(u)}{r^2} + \frac{p(u)}{r^3} + \sum_{n=1}^{\infty}\frac{m^n}{2r^{n+4}}\int_{-\infty}^{u} p(u_0)\left[(n+3)f_n'\left(\frac{u-u_0}{2r}\right) + \frac{u-u_0}{2r}f_n''\left(\frac{u-u_0}{2r}\right)\right]\mathrm{d}u_0\right),$$
$$r > 2m \tag{28}$$

$$\phi_0 = -\frac{p(u)}{r^3} - \sum_{n=1}^{\infty}\frac{m^n}{2r^{n+4}}\int_{-\infty}^{u} p(u_0)v_n'\left(\frac{u-u_0}{2r}\right)\mathrm{d}u_0, \qquad r > 2m. \tag{29}$$

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